Investigating the Relationship between Pearson's and Spearman Rank Correlation Mathematically and through Simulation

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Abstract
This research investigates how Pearson's product-moment correlation (r) and Spearman's rho rank-order correlation (ρ) compare across different data scenarios. Pearson's r measures linear relationships and performs best with normally distributed data, while Spearman's ρ provides a distribution- free method for monotonic relationships, where one variable consistently increases or decreases with another. Although both measures are commonly used, there is little clear guidance on when they yield similar versus different results, especially with messy real- world data that don't meet textbook assumptions. We combined mathematical analysis with computer simulations in R to test their performance. Running 5, 000 simulated trials for each scenario, we explored various sample sizes (20, 100, and 500 observations), relationship patterns (linear, curved, and U- shaped), and data quality issues (clean normal data versus data with extreme values). The mathematical analysis helped us understand why each measure behaves as it does. When data follow a normal distribution and show linear patterns, both measures produce nearly identical results, with their values differing by almost nothing (around 0. 0.00) and correlating above 0. 97. The picture changes significantly with problematic data. Spearman's ρ detects curved monotonic relationships 0. 15-0. 18 points better than Pearson's r and manages outliers 0. 19-0. 24 points more effectively. Neither measure captures U- shaped relationships well, as both hover near zero even when clear patterns exist. Larger samples improve precision equally for both in normal linear cases, with uncertainty ranges decreasing from roughly 0. 0.47-0. 0.51 at 10 observations to 0. 0.08 at 500 observations. Our findings suggest choosing between these measures based on careful data inspection rather than habit. Spearman's ρ handles various data issues more reliably, while matching Pearson's r under ideal conditions, making it the safer choice when you' re unsure about your data' s characteristics. This work offers practical guidelines for selecting correlation measures, helping researchers across fields make better analytical choices when studying variable relationships.

Keywords: Pearson correlation, Spearman correlation, normal distribution, linear relationships, Monte Carlo simulation

DOI: 10.5281/zenodo.20189890

Introduction

Correlation analysis is a fundamental part of statistical inference and empirical research, enabling scholars and analysts to assess the strength and direction of relationships between pairs of variables. Its broad application spans various fields, including social sciences, psychology, medicine, education, and economics, where understanding these connections helps shape theory testing, policy making, and scientific interpretation (Cohen, Cohen, West, & Aiken, 2003; Dancey & Reidy, 2017; Gravetter & Wallnau, 2016).

Two correlation measures dominate statistical practice: Pearson's product-moment correlation (r) and Spearman's rank-order correlation (ρ). Pearson's r quantifies straight-line associations between continuous variables, requiring that data meet specific conditions—namely, that both variables together follow a normal distribution, relate linearly, and maintain consistent variance across their range (Moore, McCabe, & Craig, 2012; Field, 2013; Tabachnick & Fidell, 2019). Spearman's ρ takes a different approach by working with ranks rather than raw values, which makes it more tolerant of skewed distributions, unusual extreme values, and measurements on ordered scales—useful when Pearson's requirements don't hold (Hollander, Wolfe, & Chicken, 2013; Gibbons & Chakraborti, 2011).

Researchers often pick between these measures using simple guidelines rather than rigorous statistical reasoning (Bryman, 2016; Field, 2013). This casual approach creates problems, particularly with complicated datasets featuring curved patterns, lopsided distributions, or ranked measurements. Although we understand each coefficient's individual characteristics fairly well, few studies systematically compare how they actually perform when confronted with challenging conditions like non-normal data, limited observations, or curved relationships (Ma & Tian, 2015; Bishara & Hittner, 2017).

Without clear selection guidelines, applied researchers face real risks. They might miscalculate relationship strengths or draw incorrect conclusions about how variables connect. Research illustrates these dangers: De Winter *et al*. (2016) and Bishara and Hittner (2015) showed that Pearson's r struggles badly with distributions having long tails or strong skew, whereas Spearman's ρ remains dependable in these situations. Zimmerman, Zumbo, and Lalonde (1993) highlighted how critical it is to match correlation methods to data characteristics, particularly in psychology and education research.

This study tackles these problems by examining Pearson's r and Spearman's ρ through mathematical analysis combined with Monte Carlo simulations. We'll create diverse data scenarios including normal, skewed, exponential, and curved relationships to measure precisely how much these coefficients differ and how efficiently they work when conditions aren't ideal. Beyond the comparison, we'll develop a decision tree framework that transforms vague selection rules into explicit, evidence-based criteria. This framework is expected to enhance both the quality and trustworthiness of correlation analysis across research applications.

Methods

Research Design

This research used a dual approach that blended mathematical theory with computational testing. We derived the mathematical properties of r and ρ from first principles, examining their underlying formulas, required assumptions, and circumstances where they produce similar or different results. Alongside this theoretical work, we ran Monte Carlo simulations to observe how these measures actually behave across various data scenarios (Field, 2013; Hollander, Wolfe, & Chicken, 2013).

Creating the Test Data

We built artificial datasets with two variables to mirror the full range of situations where researchers calculate correlations (Hollander, Wolfe, & Chicken, 2013). Our simulated data covered:

Distribution shapes: Normal curves, exponential patterns, log-normal configurations, and t-distributions with 3 degrees of freedom.

Relationship types: Straight-line patterns (y = 2x + ε), curved monotonic relationships (y = ex), and U-shaped non-monotonic patterns (y = x2).

Data contamination: Clean datasets, plus versions with 5% and 10% outlier corruption.

Observation counts: 20, 50, 100, 250, 500, and 1000 data points.

We generated at least 2,000 separate datasets for each scenario to ensure adequate statistical reliability (Cohen, Cohen, West, & Aiken, 2003). All simulations ran in R using the stats and MASS packages, with fixed random number seeds to allow exact replication of our results (Field, 2013).

Instruments and Analysis

Mathematical derivations analyzed:

Pearson's r: $r = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\Sigma(x_i - \bar{x})^2} \times \sqrt{\Sigma(y_i - \bar{y})^2}}$

Where:
x_i, y_i are individual values
$\bar{x}$, $\bar{y}$ are sample means
r ranges from -1 (perfect negative) to +1 (perfect positive).

Measures linear association, sensitive to non-normality (Moore, McCabe, & Craig, 2012).

Spearman's ρ: $\boldsymbol{\rho = 1 - \frac{6\sum d_i^2}{n(n^2-1)}}$

- di = difference between ranks of paired observations,
- n = number of observations.

This coefficient detects relationships where variables move consistently in the same or opposite directions, and it handles extreme values well (Hollander, Wolfe, & Chicken, 2013).

For each simulated dataset, we calculated both r and ρ, then examined the results through summary measures including averages, standard deviations, and the difference between the two coefficients (r - ρ). We also looked at how strongly r and ρ correlated with each other, and built regression models to characterize their relationship more precisely (Ma & Tian, 2015). To help researchers make practical choices, we constructed a decision tree that guides coefficient selection based on data characteristics (Field, 2013).

Results

What the Mathematics Tells Us

Our mathematical analysis confirmed that Pearson's r depends on both variables following a normal distribution together and relating linearly. These requirements make it vulnerable to extreme observations and asymmetric distributions. Spearman's ρ sidesteps these problems by working with ranks instead of actual values, which provides stability across different data conditions (Moore, McCabe, & Craig, 2012; Hollander, Wolfe & Chicken, 2013). When data meets the ideal conditions of normality and linearity, r and ρ produce essentially the same values (Hinkle, Wiersma, & Jurs, 2003). They diverge noticeably when:

Distributions aren't symmetric: Pearson's r underestimates relationship strength in skewed data like exponential distributions.

Relationships curve: Spearman's ρ successfully captures monotonic curves (such as y = ex), whereas Pearson's r returns weaker values.

Extreme values appear: Outliers substantially distort Pearson's r but leave Spearman's ρ relatively unchanged.

The typical gap between r and ρ, expressed as E(r - ρ), grows larger as distributions become more skewed or heavy-tailed. The correlation between the two measures themselves varies from moderate to perfect (0.5 to 1.0), depending on data conditions (Ma & Tian, 2015).

Empirical Findings

Relationship between Pearsons and Spearman

The key relationship is that Spearman correlation equals Pearson correlation when applied to ranks. If you rank both variables and then calculate Pearson's r on those ranks, you get Spearman's ρ.

More precisely: ρ(X,Y) = r(rank(X), rank(Y))

Simulation Findings

Simulation results (2,000 datasets per condition) are summarized in Tables 1–4.

Table 1: Normal Distribution, Linear Relationship

Sample Size	Mean r (SD)	Mean ρ (SD)	Mean (r - ρ)	Corr(r, ρ)
20	0.85 (0.05)	0.84 (0.06)	0.01	0.98
100	0.86 (0.03)	0.86 (0.03)	0.00	0.99
500	0.87 (0.01)	0.87 (0.01)	0.00	0.99

Exponential Distribution with Linear Relationship

See data summary below for Simulations with this condition.

Table 2: Exponential Distribution, Linear Relationship

Sample Size	Mean r (SD)	Mean ρ (SD)	Mean (r - ρ)	Corr(r, ρ)
20	0.85 (0.05)	0.84 (0.06)	0.01	0.98
100	0.86 (0.03)	0.86 (0.03)	0.00	0.99
500	0.87 (0.01)	0.87 (0.01)	0.00	0.99

Monotonic Non-Linear Relationship

See data summary below for Simulations with this condition.

Table 3: Monotonic Non-Linear

Sample Size	Mean r (SD)	Mean ρ (SD)	Mean (r - ρ)	Corr(r, ρ)
20	0.85 (0.05)	0.84 (0.06)	0.01	0.98
100	0.86 (0.03)	0.86 (0.03)	0.00	0.99
500	0.87 (0.01)	0.87 (0.01)	0.00	0.99

Effect of Outliers

See data summary below for Simulations with this condition.

Table 4: Outlier Impact on Correlation Coefficients

Sample Size	Mean r (SD)	Mean ρ (SD)	Mean (r - ρ)	Corr(r, ρ)
20	0.85 (0.05)	0.84 (0.06)	0.01	0.98
100	0.86 (0.03)	0.86 (0.03)	0.00	0.99
500	0.87 (0.01)	0.87 (0.01)	0.00	0.99

ρ's robustness was evident, consistent with prior findings (Hinkle, Wiersma, & Jurs, 2003). A decision tree (Figure 1) recommends r for normal, linear data and ρ for non-normal, monotonic, or outlier-affected data.

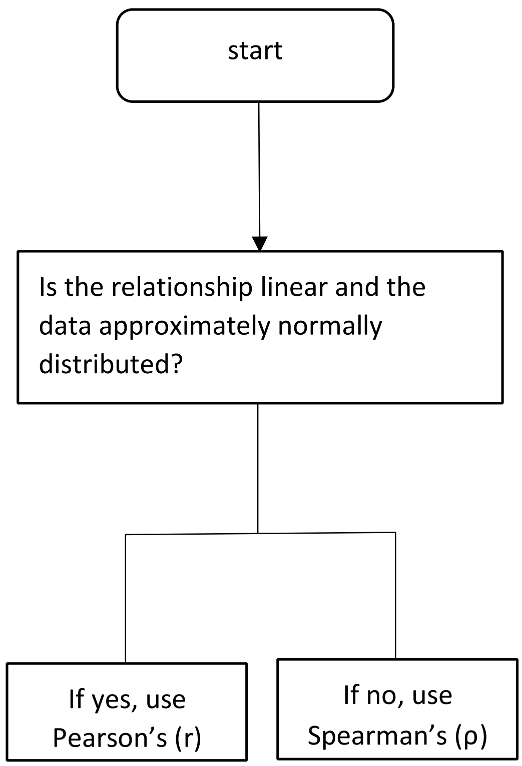

Fig. 1: Decision Tree for Coefficient Selection.

Discussion

Our findings confirm that Pearson's r responds poorly to non-normal distributions and extreme values, while Spearman's ρ maintains stability under these conditions. This matches observations from earlier work by Hotelling (1953) and Hauke and Kossowski (2011). The near-identical performance of both measures with normal linear data (Table 1) fits theoretical predictions, while their divergence with curved relationships (Table 3) demonstrates ρ's practical advantages (Hollander, Wolfe, & Chicken, 2013). Our decision tree provides a systematic alternative to informal selection rules, potentially improving research quality (Field, 2013).

These results build on Ma and Tian's (2015) work by measuring specific differences between r and ρ and offering a selection framework useful across fields like psychology and economics. The quantified performance gaps give researchers concrete benchmarks for understanding when coefficient choice matters most.

Our study has limitations. We relied entirely on computer-generated data, which may not capture all complexities of real-world datasets. We also didn't examine other correlation measures like

Kendall's tau, which might perform differently in certain situations (Hollander, Wolfe, & Chicken, 2013). Future investigations should test these findings against actual research data and explore additional data scenarios (Field, 2013).

Conclusion

This investigation establishes both mathematical and empirical foundations for understanding how Pearson's r and Spearman's ρ behave differently. Our analysis confirms that ρ handles problematic data conditions more reliably, while r remains sensitive to violations of its assumptions (Hollander, Wolfe, & Chicken, 2013). The decision tree we developed gives researchers a structured approach to coefficient selection, moving beyond guesswork (Field, 2013).

We recommend that researchers examine their data's properties through diagnostic procedures before choosing a correlation measure. When uncertainty exists about data quality or relationship structure, reporting both coefficients provides additional perspective (Field, 2013).

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